MULTIMEDIA UNIVERSITY

STUDENT ID NO

MULTIMEDIA UNIVERSITY

FINAL EXAMINATION

TRIMESTER 2, 2019/2020

EME4116 – COMPUTATIONAL FLUID DYNAMICS

(ME)

9 MARCH 2020

2.30 p.m. - 4.30 p.m.

(2 Hours)

INSTRUCTIONS TO STUDENTS

1. This Question Paper consists of seven pages including the cover page and Appendix.
2. Answer ALL questions. Each question carries 25 Marks and the distribution of the Marks for each question is given in brackets [ ].
3. Write all your answers in the Answer Booklet provided.

Question 1

(a) Consider the steady, inviscid equations for supersonic flow slightly perturbed by a thin body, the small disturbance potential equation may be written as the system:

$$(1 - M_\infty{}^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

where M_∞ is the freestream Mach number.

(i) Classify the behavior of the system of PDEs, using eigenvalue method.

[6 marks]

(ii) From part (i), determine the directions of the two characteristics.

[2 marks]

(b) Show that the following representation has a truncation error of O (Δx^2).

$$\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j} = \frac{2u_{i,j} - 5u_{i-1,j} + 4u_{i-2,j} - u_{i-3,j}}{\Delta x^2}$$

[7 marks]

(c) Suppose the Lax scheme is applied to solve the linear wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

with c given as 0.75, and is subjected to the initial condition :

$u\,(x, 0) = 2\sin 2\pi x,\ \ 0 \le x \le 1.$

Calculate the amplitude errors after 10 time steps for $\nu = 0.25$ and

$\Delta x = 0.02$ **[10 marks]**

Note: 1. Amplification factor for Lax scheme is, $G = \cos\beta - i\nu\sin\beta$ where

$\beta = k_m \Delta x,\ \nu = \dfrac{c\Delta t}{\Delta x}$.

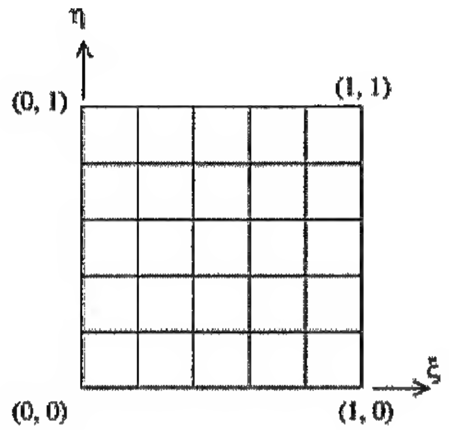
2. Exact elemental solution, $u = e^{ik_m(x-ct)}$.

Continued…..

Question 2

(a) A trapezoidal region $(x,\ y)$ is mapped to a computational plane (ξ,η) corresponding to ξ,η as shown in Figure Q2. If the Cartesian coordinates for the four points on the trapezoid are A (0, 0), B(1, 0), C(1, 5), and D (0, 2) respectively, devise the transformation for $x\ (\xi,\eta)$ and $y\ (\xi,\eta)$.

[6 marks]



(b) The 2D physical space (x,y) is transformed into the computational space (ξ,η) by using the transformation,

$$x = \xi$$

$$y = \eta(\xi + 2 - 2\xi^2)$$

(i) Find the Jacobian of this transformation. **[6 marks]**

(ii) By using the Jacobian in part (i), determine the continuity equation of an incompressible flow transformed into computational space.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Display the result in strong conservation form. The transformed equation should contain ξ and η as the only independent variables.

[13 marks]

Note: To recast $\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$ in computational space in the form,

$$\frac{\partial U_1}{\partial t} + \frac{\partial F_1}{\partial \xi} + \frac{\partial G_1}{\partial \eta} = 0,$$

$$U_1 = JU, \quad F_1 = JF\frac{\partial \xi}{\partial x} + JG\,\frac{\partial \xi}{\partial y}, \; G_1 = JF\frac{\partial \eta}{\partial x} + JG\,\frac{\partial \eta}{\partial y}.$$

Continued.....

Question 3

Consider a thermally conducting fluid contained between two large parallel walls separated by a distance L. Assume that the fluid temperature is initially constant everywhere at T=0 °C, in equilibrium with both walls at $T_{w1} = T_{w2} = 0$ °C. At time

$t \geq 0$, the temperature at the lower wall, T_{w1} is impulsively increased to 50 °C, causing a transient change to the fluid temperature, $T(y, t)$. The initial and boundary conditions are stated as:

$$T(y, 0) = 0,$$

$$T(0, t) = 50, \text{ and } T(L, t) = 0 .$$

Take the fluid thermal diffusivity, $\alpha = 0.001$ m²/s, and $L = 1$ m.

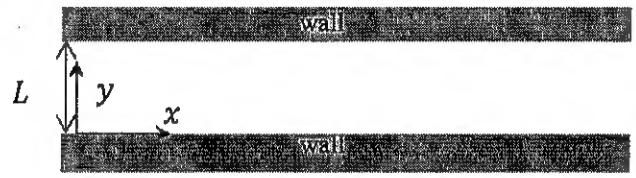


Figure Q3

Neglect any possible effect due to natural convection, the governing heat equation for this problem may be stated as,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \quad .$$

(a) Write down the finite difference equation for the heat equation, by using forward difference representation for the time derivative and a central difference representation for the space derivative. Use Δy to denote adjacent nodes along y direction. **[3 marks]**

(b) Referring to the finite difference equation in part (a), *calculate* the temperature at point y =0.1 m for two time steps. Use Δy =0.1 m, and specify r =1/6. **[12 marks]**

(c) Determine the amplification factor, G for the finite difference equation in (a) in terms of r and β. **[4 marks]**

(d) Find $|G_e| - |G|$ and determine the optimum r that minimises the error.

Hint: You may need to expand G and G_e in terms of β until β^4. **[6 marks]**

Note: 1. $\beta = k_m \Delta y, \ r = \dfrac{\alpha \Delta t}{(\Delta y)^2}$

2. The exact amplification factor, G_e in terms of r and β is,

$$G_e = e^{-\alpha k_m{}^2 \Delta t} = e^{-r\beta^2} .$$

Continued.....

Question 4

Consider a non-linear inviscid Burgers equation given as,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,.$$

The equation can be recast as:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$$

where $F = u^2/2$.

(a) Starting from time differencing by putting,

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2}\left[\left(\frac{\partial u}{\partial t}\right)^n + \left(\frac{\partial u}{\partial t}\right)^{n+1}\right]$$

and incorporating Beam and Warming (1976) method to put,

$$F^{n+1} = F^n + A^n\left(u_j^{n+1} - u_j^n\right)$$

where $A = \frac{\partial F}{\partial u}$, *show* that,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2}\left\{2\left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}\left[A^n\left(u_j^{n+1} - u_j^n\right)\right]\right\}.$$ **[5 marks]**

(b) By replacing the x derivatives by central difference in part (a), *derive* the Beam and Warming implicit scheme given as,

$$-\frac{\Delta t A_{j-1}^n}{4\Delta x}u_{j-1}^{n+1} + u_j^{n+1} + \frac{\Delta t A_{j+1}^n}{4\Delta x}u_{j+1}^{n+1}$$
$$= -\frac{\Delta t}{\Delta x}\left(\frac{F_{j+1}^n - F_{j-1}^n}{2}\right) - \frac{\Delta t A_{j-1}^n}{4\Delta x}u_{j-1}^n + u_j^n + \frac{\Delta t A_{j+1}^n}{4\Delta x}u_{j+1}^n$$

[4 marks]

(c) The Burgers equation is subjected to the initial and boundary conditions:

$u(x,0) = -x,\ 0 \le x \le 1,$

$u(0,t) = 0,\ u(1,t) = -1,\ t > 0\,.$

By applying the Beam and Warming implicit scheme, *outline* an algorithm to compute u by taking $\Delta x = 0.02$.

[16 marks]

Continued....

APPENDIX

A1. Taylor's Series of Expansion

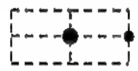
$$u(x+\Delta x, y) = u(x,y) + \frac{\Delta x}{1!}\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots$$

$$u(x-\Delta x, y) = u(x,y) - \frac{\Delta x}{1!}\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4} + \cdots$$

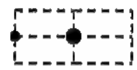
$$u(x, y+\Delta y) = u(x,y) + \frac{\Delta y}{1!}\frac{\partial u}{\partial y} + \frac{(\Delta y)^2}{2!}\frac{\partial^2 u}{\partial y^2} + \frac{(\Delta y)^3}{3!}\frac{\partial^3 u}{\partial y^3} + \frac{(\Delta y)^4}{4!}\frac{\partial^4 u}{\partial y^4} + \cdots$$

A2. Derivatives of the Finite Difference

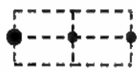
$$\left.\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$



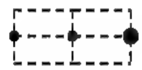
$$\left.\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x)$$



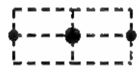
$$\left.\frac{\partial u}{\partial x}\right)_{i,j} = \frac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2\Delta x} + O[(\Delta x)^2]$$



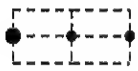
$$\left.\frac{\partial u}{\partial x}\right)_{i,j} = \frac{3u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{2\Delta x} + O[(\Delta x)^2]$$



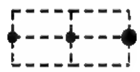
$$\left.\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O[(\Delta x)^2]$$



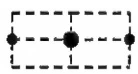
$$\left.\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i,j} - 2u_{i+1,j} + u_{i+2,j}}{(\Delta x)^2} + O(\Delta x)$$



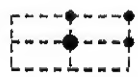
$$\left.\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i,j} - 2u_{i-1,j} + u_{i-2,j}}{(\Delta x)^2} + O(\Delta x)$$



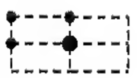
$$\left.\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O[(\Delta x)^2]$$



$$\left.\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(u_{i+1,j+1} - u_{i+1,j}\right) - \left(u_{i,j+1} - u_{i,j}\right)}{\Delta x \Delta y} + O(\Delta x, \Delta y)$$



$$\left.\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(u_{i,j+1} - u_{i,j}\right) - \left(u_{i-1,j+1} - u_{i-1,j}\right)}{\Delta x \Delta y} + O(\Delta x, \Delta y)$$



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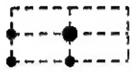
$$\left.\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(u_{i,j} - u_{i,j-1}\right) - \left(u_{i-1,j} - u_{i-1,j-1}\right)}{\Delta x \Delta y} + O(\Delta x, \Delta y)$$



$$\left.\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(u_{i+1,j} - u_{i+1,j-1}\right) - \left(u_{i,j} - u_{i,j-1}\right)}{\Delta x \Delta y} + O(\Delta x, \Delta y)$$

End of Paper.